Further Calculus VI Cheat Sheet

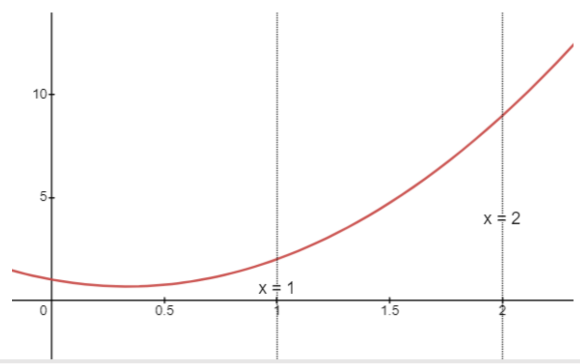
Mean Value of a Function

Integration can be used to find the mean value of a function between two points. By integrating, all the values that the function takes within the given interval are summed, rather than taking the beginning and end values and finding their midpoint. This gives a better measurement of the mean value that the function takes between the two points.

The following formula gives the mean value of a function $f(x)$ between $x = a$ and $x = b$:

$$\text{Mean value} = \frac{\int_a^b f(x)dx}{b-a}$$

Example 1: Find the mean value of the function $f(x) = 3x^2 - 2x + 1$ between the points $x = 1$ and $x = 2$.

Use the above formula for the mean value with $f(x) = 3x^2 - 2x + 1$, and with $a = 1, b = 2$.	$\frac{\int_1^2 3x^2 - 2x + 1\, dx}{2-1} = [x^3 - x^2 + x]_1^2 = \left((6) - (1)\right) = 5$ $\therefore$ the mean value is 5
This is the graph of $y = f(x)$. The mean value considers all the values that the function takes between each limit. As $f(2) = 6$, and the graph has a rapidly increasing gradient, intuitively, the values closer to the right are larger so contribute more the total. Thus the average is close to 6.	

Example 2: Consider a function $g(x)$. Suppose its mean value between the points $x = x_1$ and $x = x_2$ is G.

a.) Prove that the mean value of $g(x) + 2$ between $x = x_1$ and $x = x_2$ is $G + 2$.

b.) Generalise this result for $g(x) + n$, where n is any number.

a.) Find an expression for the mean value of $g(x)$ as an integral, and then integrate $g(x) + 2$ by splitting into two, evaluating $\int_{x_1}^{x_2} 2\, dx$ explicitly. Use the result for the mean value for $g(x)$ to arrive at the result. Intuitively, if a function is translated upward by 2, every value that the function takes is increases by 2, and therefore the mean will also be increased by 2.	The mean value of $g(x)$ between the given limits is $\frac{\int_{x_1}^{x_2} g(x)dx}{x_2 - x_1} = G$ The mean value of $g(x) + 2$ is given by $\frac{\int_{x_1}^{x_2} g(x) + 2\, dx}{x_2 - x_1} = \frac{\int_{x_1}^{x_2} g(x)dx + \int_{x_1}^{x_2} 2\, dx}{x_2 - x_1}$ $\int_{x_1}^{x_2} 2\, dx = 2(x_2 - x_1)$ $\therefore \frac{\int_{x_1}^{x_2} g(x) + 2\, dx}{x_2 - x_1} = \frac{\int_{x_1}^{x_2} g(x)dx + 2(x_2 - x_1)}{x_2 - x_1} = \frac{\int_{x_1}^{x_2} g(x)dx}{x_2 - x_1} + 2 = G + 2$
b.) The general case follows as the integral of any given number n will be the length of the interval multiplied by n.	In general, for any number n, $\frac{\int_{x_1}^{x_2} g(x) + n\, dx}{x_2 - x_1} = \frac{\int_{x_1}^{x_2} g(x)dx + \int_{x_1}^{x_2} n\, dx}{x_2 - x_1} = \frac{\int_{x_1}^{x_2} g(x)dx + n(x_2 - x_1)}{x_2 - x_1} = G + n$ $\therefore$ The mean of $g(x) + n$ is $G + n$.

Reduction Formulae (A Level Only)

Some integrals require the use of integration by parts several times before they can be evaluated, for example it is used twice with $\int x^2 e^x dx$. When the power of x is higher, the higher the number of times "by parts" needs to be applied. A recursive sequence can be formed by repeated integration by parts. Denoting the integral as $I_n = \int x^n e^x dx$, the following result holds:

$$I_n = x^n e^x - \int n x^{n-1} e^x dx = x^n e^x - nI_{n-1}$$

This is an example of a reduction formula. By evaluating I_0, I_n can be evaluated without further integration by using the reduction formula. This type of formula can be found for many integrals. In finding a reduction formula, the aim is to manipulate the expressions into a form similar to the original integral – so that it can be related recursively. Often this will involve initially integrating by parts.

Example 3: **a.)** If $I_n = \int_0^{\frac{\pi}{4}} x^n \cos(2x)\, dx$, use integration by parts to derive a reduction formula for I_n.

b.) Use this formula to find exact value of I_4.

a.) Perform integration by parts twice since there is a $\cos(x)$ in I_n. Differentiating the x^n will allow for terms of the kind I_{n-k} to be found, since n appears only as the power of x in I_n. Evaluate the integrals, and group together the terms to reach the final formula, taking note of the expression fitting the form of I_{n-2}.	Let $u = x^n, \quad \frac{dv}{dx} = \cos(2x) \Rightarrow \frac{du}{dx} = nx^{n-1}, v = \frac{1}{2}\sin(2x)$ Using $\int u dv = uv - \int v\, du$ gives $\int_0^{\frac{\pi}{4}} x^n \cos(2x)\, dx = \left[\frac{1}{2}x^n \sin(2x)\right]_0^{\frac{\pi}{4}} - \frac{n}{2}\int_0^{\frac{\pi}{4}} x^{n-1}\sin(2x) = \frac{1}{2}\left(\frac{\pi}{4}\right)^n - \frac{n}{2}\int_0^{\frac{\pi}{4}} x^{n-1}\sin(2x)\, dx$ Perform integration by parts again: $u = x^{n-1}, \quad \frac{dv}{dx} = \sin(2x) \Rightarrow \frac{du}{dx} = (n-1)x^{n-2}, v = -\frac{1}{2}\cos(2x)$ $\therefore \int_0^{\frac{\pi}{4}} x^{n-1}\sin(2x)\, dx = \left[-\frac{1}{2}x^{n-1}\cos(2x)\right]_0^{\frac{\pi}{4}} + \frac{n-1}{2}\int_0^{\frac{\pi}{4}} x^{n-2}\cos(2x) = 0 + \frac{n-1}{2}\int_0^{\frac{\pi}{4}} x^{n-2}\cos(2x)\, dx$ $\int_0^{\frac{\pi}{4}} x^{n-2}\cos(2x) = I_{n-2} \therefore I_n = \frac{1}{2}\left(\frac{\pi}{4}\right)^n - \frac{n(n-1)}{4} I_{n-2}$
b.) Begin by finding I_0, and then use the reduction formula twice to find I_4.	$I_0 = \int_0^{\frac{\pi}{4}} \cos(2x)\, dx = \frac{1}{2}[\sin(2x)]_0^{\frac{\pi}{4}} = \frac{1}{2}$ $I_2 = \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \frac{2(2-1)}{4} I_0 = \frac{\pi^2}{32} - \frac{1}{4}$ $I_4 = \frac{1}{2}\left(\frac{\pi}{4}\right)^4 - \frac{4(4-1)}{4} I_2 = \frac{\pi^4}{512} - 3I_2 = \frac{\pi^4}{512} - \frac{3\pi^2}{32} + \frac{3}{4}$

Reduction formulae may also be found when the integrand involves trigonometric functions.

Example 4: Find a reduction formula for $I_n = \int_0^{\ln(4)} \tanh^n(x) dx$.

Write $\tanh^n(x)$ as $\tanh^2(x)\tanh^{n-2}(x)$. Use the identity $\tanh^2(x) = 1 - \operatorname{sech}^2(x)$ to rewrite the integrand and split it into two integrals. Integrate the second by reversing the chain rule, using the result that the derivative of $\tanh(x)$ is $\operatorname{sech}^2(x)$. The first integral is the same as I_n, but with the power $n - 2$, and so this is just I_{n-2}. Combine the two terms to arrive at the reduction formula.	$\tanh^n(x) = \tanh^{n-2}(x)\tanh^2(x) = \tanh^{n-2}(x)(1 - \operatorname{sech}^2(x))$ $\therefore \int_0^{\ln(4)} \tanh^n(x)\, dx = \int_0^{\ln(4)} \tanh^{n-2}(x) - \tanh^{n-2}\operatorname{sech}^2(x)\, dx$ $\frac{d}{dx}\tanh^{n-1}(x) = (n-1)\tanh^{n-2}\operatorname{sech}^2(x) \Rightarrow \frac{1}{n-1}\tanh^{n-1}(x) = \int \tanh^{n-2}(x)\operatorname{sech}^2(x)\, dx$ $\therefore \int_0^{\ln(4)} \tanh^n(x)\, dx = \int_0^{\ln(4)} \tanh^{n-2}(x)\, dx - \frac{1}{n-1}[\tanh^{n-1}(x)]_0^{\ln(4)}$ $= \mathrm{I}_{n-2} - \frac{1}{n-1}\left(\frac{15}{17}\right)^{n-1}$ $\therefore \mathrm{I}_n = \mathrm{I}_{n-2} - \frac{1}{n-1}\left(\frac{15}{17}\right)^{n-1}$